5-2 Videos Guide

5-2a

Expressions below are given in $\mathbb{R}^3$. Analogous expressions exist for functions in $\mathbb{R}^2$ by simply leaving off the z- or $\mathbf{k}$-component

- Line integral of a scalar function f with respect to arc length: over a curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$
  - $\int_C f(x,y,z)\, ds = \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
    $= \int_a^b f\big(\mathbf{r}(t)\big)|\mathbf{r}'(t)|\, dt$
  - Note that $ds = |\mathbf{r}'(t)|\, dt$
- Line integral with respect to x
  - $\int_C f(x,y,z)\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt$ (likewise for y and z)

Exercises:

5-2b

- Evaluate the line integral, where C is the given curve.
  - $\int_C xe^y\, ds$, C is the line segment from $(2, 0)$ to $(5, 4)$

5-2c

- $\int_C x^2\, dx + y^2\, dy$, C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ followed by the line segment from $(0, 2)$ to $(4, 3)$

5-2d

Expressions below are given in $\mathbb{R}^3$. Analogous expressions exist for functions in $\mathbb{R}^2$ by simply leaving off the z- or $\mathbf{k}$-component

- Line integral of a vector field $\mathbf{F}(x,y,z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{j}$ over a curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$
  - $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\, dt = \int_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_C P\, dx + Q\, dy + R\, dz$
  - $\left(\text{This really means } \int_C P\, dx + \int_C Q\, dy + \int_C R\, dz\right)$
  - Note that $d\mathbf{r} = \mathbf{r}'(t)dt$
  - Also note that we generally parameterize P, Q, and R
- Final notes about orientation
  - $\int_{-C} f(x,y,z)\, dx = -\int_C f(x,y,z)\, dx$ (and likewise for y and z)
    and $\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$
    but $\int_{-C} f(x,y,z)\, ds = \int_C f(x,y,z)\, ds$

Exercises:

5-2e

- Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.
  $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + xy\,\mathbf{j} + (y + z)\,\mathbf{k}$,
  $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - 2t\,\mathbf{k},\ 0 \le t \le 2$

5-2f

- Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.